Eisenstein series in the Siegel setting

Recall: fix $\underline{g \geq 1}$, $\rho : GL_g(\mathbb{C}) \to GL(V_\rho)$ (maybe irreducible?)

$\Gamma_g = Sp_{2g}(\mathbb{Z})$. A Siegel mod form of weight ρ on Γ_g

is holom $f : \mathcal{H}_g \longrightarrow V_\rho$ such that

$$f\left((a\tau+b)(c\tau+d)^{-1}\right) = \rho(c\tau+d)f(\tau) \quad \forall \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \in \Gamma_g$$

$$\forall \boxed{\tau} \in \mathcal{H}_g = \{\tau \in M_g(\mathbb{C}) \; {}^t\tau = \tau, \; \mathrm{Im}(\tau) > 0\}$$

(if $g=1$, need an extra growth condition)

If $\tau' \in \mathcal{H}_{g-1}$ and $t \in \mathbb{R}_{>0}$, then

$$\begin{bmatrix} \tau' & 0 \\ 0 & it \end{bmatrix} \in \mathcal{H}_g$$

Given $f \in M_\rho(\Gamma_g)$, define $\Phi f : \mathcal{H}_{g-1} \to V_{\rho'}$

$$(\Phi f)(\tau') = \lim_{t \to \infty} f\left(\begin{bmatrix} \tau' & 0 \\ 0 & it \end{bmatrix}\right)$$

Outcome: $\Phi : M_\rho(\Gamma_g) \longrightarrow M_{\rho'}(\Gamma_{g-1})$

ρ highest wt $\lambda_1 \geq \cdots \geq \lambda_g$ $\rho : GL_g(\mathbb{C}) \to GL(V_\rho)$

$\rightsquigarrow$

ρ' highest wt $\lambda_1 \geq \cdots \geq \lambda_{g-1}$ $\rho' : GL_{g-1}(\mathbb{C}) \to GL(V_{\rho'})$

if $f(\tau) = \sum_{\substack{n \geq 0 \\ n \in \mathcal{S}_g}} a(n)\, e^{2\pi i \operatorname{Tr}(n\tau)}$ $\quad a(n) \in V_\rho$

$n > 0$

$\mathcal{S}_g = \left\{ n \in M_g\left(\tfrac{1}{2}\mathbb{Z}\right) \,\middle|\, {}^t n = n,\ n_{ii} \in \mathbb{Z}\ \forall i \right\}$

then

$$(\Phi f)(\tau') = \sum_{\boxed{\substack{n' \geq 0 \\ n' \in \mathcal{S}_{g-1}}}} a\left(\begin{bmatrix} n' & 0 \\ 0 & 0 \end{bmatrix}\right) e^{2\pi i \operatorname{Tr}(n'\tau')}$$

Siegel Φ-operator.

$$\Phi : M_\rho(\Gamma_g) \to M_{\rho'}(\Gamma_{g-1})$$

$\operatorname{Ker} \Phi =: S_\rho(\Gamma_g)$ cusp forms of weight ρ for Γ_g.

if $g = 1$: $\Phi(f) = a(0)$
$f \in M_k(\Gamma_1)$

$\Phi : M_\rho(\Gamma_g) \to M_{\rho'}(\Gamma_{g-1})$ can iterate $g-r$ times

$$\boxed{\Phi^{(g-r)} : M_\rho(\Gamma_g) \to M_\rho(\Gamma_r)}$$

$S_\rho(\Gamma_g) \subseteq M_\rho(\Gamma_g)$

$f, h \in M_k(\Gamma_1)$

$\tau = x + iy$

<u>Petersson inner product</u>:

$g=1$ $\quad \langle f, h \rangle = \int_{\mathcal{F}} f(\tau)\overline{h(\tau)} \boxed{y^{k-2}\, dx\, dy}$

if at least one of f, h is a cusp form

$\Downarrow$

absolute convergence

$$= \int f(\tau)\overline{h(\tau)} \left(\mathrm{Im}(\tau)\right)^k \frac{dx\,dy}{y^2}$$

$\frac{dx\,dy}{y^2} = d\tau$

$g>1$:

$$\langle f,h\rangle = \int_{\Gamma_g\backslash\mathcal{H}_g} \Big(\rho(\operatorname{Im}\tau)\, f(\tau), h(\tau)\Big)\, d\tau$$

$$\boxed{d\tau = \frac{1}{\det^{g+1}(y)} \prod_{i\leq j} dx_{ij}\, dy_{ij}}$$

$f, h \in M_\rho(\Gamma_g)$

Have $\rho : GL_g(\mathbb{C}) \to GL_g(V_\rho)$

$\rho : GL_g(\mathbb{C}) \to GL(V_\rho)$

V_ρ fin dim.

$\exists$ Hermitian inner product $(\cdot,\cdot)$ on V_ρ s.t.

$$\big(\rho(\gamma)v_1, v_2\big) = \big(v_1, \overline{\rho(\gamma^t)}\, v_2\big)$$

$\tau \in \mathcal{H}_g$

$\operatorname{Im}\tau \in GL_g(\mathbb{C})$

$\forall v_1, v_2 \in V_\rho$, $\forall \gamma \in GL_g(\mathbb{C})$

If ρ is irreducible, this $(\cdot,\cdot)$ is unique up to multiplication by scalar.

if at least one of f, h is a cusp form, the integral converges absolutely.

$\Rightarrow$ can define $S_\rho^\perp =: N_\rho$ $\qquad M_\rho = S_\rho \oplus N_\rho$

Consider scalar case: $\rho = \det^k$ $\qquad M_k$

Let $f \in S_k(\Gamma_r)$, $k > 0$ even.

$$E_{g,r,k}(f)(\tau) = \sum_{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in P_r \backslash \Gamma_g} f\left(\boxed{\left((a\tau+b)(c\tau+d)^{-1}\right)}^{*}\right) \det(c\tau+d)^{-k}$$

$$\tau = \begin{bmatrix} \tau_1 & z \\ {}^t z & \tau_2 \end{bmatrix} \in \mathcal{H}_g \quad \text{with } \tau_1 \in \mathcal{H}_r, \ \tau_2 \in \mathcal{H}_{g-r}, \quad \text{notation } \tau^* := \tau_1$$

Klingen parabolic subgroup

$$P_r = \left\{ \begin{bmatrix} a' & 0 & b' & * \\ * & u & * & * \\ c' & 0 & d' & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} \;\middle|\; \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \Gamma_r, \ u \in GL_{g-r}(\mathbb{Z}) \right\}$$

<u>Theorem</u>: $g \geq 1$, $0 \leq r \leq g$, $k > g+r+1$ even

$f \in S_k(\Gamma_r)$.

$E_{g,r,k}(f) \in M_k(\Gamma_g)$ and

$$\Phi^{(g-r)} E_{g,r,k}(f) = f.$$

why?

$r = g-1$

<u>Cor</u>: $\Phi : M_k(\Gamma_g) \to M_k(\Gamma_{g-1})$ is surjective if $k > 2g$ even.